Shokurov's Rational Connectedness Conjecture.

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Rational curves in varieties with mild sing.

Rat. curves smooth varieties.

1) X is rationally connected if for general $x, y \in X$, we have a rational curve in X passing through them.

2) $V \subseteq X$, we call a chain of curves modulo V, a union of curves C, s.t. C ∪ subset of V is connected.

3) X is rationally chain connected

modulo V. if $\forall\, x, y \in X$,

we have rational chain of curves modulo V passing through them.

if $V = \emptyset$, X is rationally chain connected.

Examples & Remarks.

- A cone over an elliptic curve.

  1st It is rationally chain connected. (lines through the vertex).

  It is not rationally connected. (the only rat. curves are the rays of the cones).

- $\mathbb{P}^1 \times C$ (elliptic curve).

  → Not rat. connected

  Not rat. chain connected.

  (all the rat. curves are fibers).

It is rat. chain connected

modulo a elliptic curve.

Cone over an elliptic, $\overset{bir}{\sim} \mathbb{P}^1 \times C$.

rationally chain connected is not preserved birationally.

- But rat. connected is bir. eq. (rat. curves get mapped to rat. curves).

- $\boxed{\text{Rational Connectedness Conjecture}}$

(X, Δ) klt log-pair.

$f: X \to \underline{S}$ projective and

$-(K_X+\Delta)$ being f-nef & f-big.

Then all fibers are rationally chain connected.

Main Theorem (of the paper).

Let (X, Δ) log pair.
$f: X \to S$ projective.
s.t. $-K_X$ is f-big and
$\mathcal{O}_X(-m(K_X+\Delta))$ is relatively
generated for some $m>0$.
Let $g: Y \to X$ any bir. morphism
and $\pi: Y \to S$, the composition.

Then every fiber of π is
rationally chain connected modulo
the inverse image of the non-klt
locus.

(X, Δ) s.t. (X, Δ) is klt

$-(K_X+\Delta)$ is nef + big

$-(K_X+\Delta)$ is ~~ample~~

implies X is rationally connected

(X,Δ) s.t. (X,Δ) is $\underline{\text{l.c}}$

$-(K_X+\Delta)$ is $\underline{\text{ample}}$

Then X is rationally connected

1st) Let's check that they are sharp.

$\boxed{\text{Ex}}$ We'll take C an elliptic curve.

$\pi: X \to C$ a $\mathbb{P}^1$-bundle over C.

- Let E a section with minimal self-intersection ($E^2 < 0$).

and F a fiber (a $\mathbb{P}^1$).

$(X, t\cancel{E})$ 1st this is not

rationally chain connected.

We will study the divisors numerically.

$$\equiv aE + bF.$$

by adjunction.

$$(K_X+E)\cdot F = \deg(K_F)$$

$$aE\cdot F = -2$$

$$a = -2.$$

$$K_X \equiv -2E + bF.$$

by adj.

$$(K_X+E)\cdot E = 0$$

$$-2E^2 + bF\cdot E + E^2 = 0$$

$b = E^2.$

$K_X \equiv -2E + E^2 F.$

E, F are effective.

$\Rightarrow$ the effective divisors

are $\equiv aE + bF$ $\boxed{\begin{matrix} a \geq 0 \\ b \geq 0 \end{matrix}}$.

$\Rightarrow$ a divisor $(aE+bF)$

is ample.

iff $(aE+bF).F > 0$

$(aE+bF).E > 0.$

i.e. $a > 0, \; b > \underbrace{-aE^2}_{\text{positive}}$

number

nef $(aE+bF).F \geq 0$

$(aE+bF).E \geq 0$.

i.e. $a \geq 0$, $b \geq -aE^2$.

as big = ample + effective

big $\Leftrightarrow$ $a>0$, $b>0$.

$$\boxed{\begin{array}{ll} \text{Ample divisor:} & \varepsilon E + bF \\ \text{Eff.} & : (a-\varepsilon)E \end{array}}$$

$(X, \underline{t}E)$.

$-(K_X + tE)$

$= (2-t)E - E^2 F$

$-(2-t)E \quad L-T.$

is big for $t<2$

nef for $1 \leq t \leq 2$

ample for $1<t<2$.

(X, tE) is klt when $t<1$

l.c when $t \leq 1$.

(X, E) $(t=1)$,

$-(K_X+\Delta)$ big and nef (but not ample)

Sing. is l.c but not k.lt.

(it is r.c.c modulo E).

(non-klt locus),

(X, E) $(t < 1.)$

$-(K_X + \Delta)$ big but not nef.

sing. $\underline{\text{k.l.t.}}$ . .

We'll study some criterions

uniruledness, r.c.c., r.c.

$\underline{\text{Prop.}}$ For a proj. variety

X, $\underline{K_X}$ is pseudo-effective

iff X is not uniruled.

$\underline{\text{Prop}}$ (Log-additivity of

Kodaira dimension).

Let Y, Z smooth proj. varieties

D an effective $\mathbb{Q}$-Divisor

H an ample Divisor.

$\psi: Y \to Z$ a morphism, s.t.

$K_Y + \Delta$ is log-canonical on the general Fiber of ψ.

K_Z is pseudo-effective.

Then for any $\varepsilon > 0$.

$$\kappa(K_Y + D + \varepsilon \psi^* H) \geq \dim Z.$$

$\boxed{\text{Lemma 1}}$

Let (X,Δ) be a proj. log pair.

$h: X \to F$, $t: F \dashrightarrow Z$, where

F and Z are projective s.t.

1) The locus of non-klt sing. of $K_X+\Delta$ does not dominate Z.

2) $K_X+\Delta$ has Kodaira dimension at least zero on a general fiber of $X \dashrightarrow Z$.

(i.e. for $g: Y \to X$, s.t.

$Y \to Z$ is a map.

$g^*(K_X+\Delta)$ has κ dim ≥ 0 on a general fiber).

3) $\kappa(K_X+\Delta) \leq 0$.

4) $\exists$ A ample on F s.t.

$h^*A \leq \Delta$.

Then Z is uniruled or a point.

Proof

K_Z is pse.

Suppose that Z is not uniruled.

Blowing up we can assume it is smooth.

Let $g: Y \to X$ be a log res. of (X, Δ) s.t. $Y \dashrightarrow Z$ is a morphism. $(Y \to X \dashrightarrow Z)$.

$\psi: Y \to Z$.

$$K_Y + \Theta = g^*(K_X + \Delta) + E.$$

where we take Θ, E effective

and E just exceptional.

$$\Gamma := \Theta + \varepsilon E'$$

ε small enough (rational)

E' is the support of the exceptional locus.

as

$$K_Y + \Gamma = K_Y + \Theta + \varepsilon E'$$
$$= \underbrace{g^*(K_X + \Delta)}_{\kappa \geq 0 \text{ in fibers}} + \underbrace{E + \varepsilon E'}_{\text{effective}}$$

$K_Y + \Gamma$ has $\kappa \geq 0$ in the gen. fiber of $\varphi : Y \to Z$.

$K_Y + \Theta$ is k.l.t in the general fiber as $K_Y + \Theta$ is $K_X + \Delta$ in the general fiber,
$K_Y + \Gamma$ is also $\underbrace{\text{k.l.t.}}_{Y \to Z}$

$\exists A$ an ample divisor on F.

$h^*A \leq \Delta$,

$$g^*h^*A \leq g^*(\Delta) \leq \Gamma$$

i.e. Γ contains the pullback of an ample divisor on F.

Take A, ample on Z

A_Z ample on Z.

$mA - t^*A_0$ on T will be globally generated. for $m >> 0$.

$H \in |mA - t^*A_0|$.

$$A \sim_{\mathbb{Q}} \frac{H}{m} + \frac{1}{m} t^* \underbrace{A_Z}_{\text{ample on } Z.}$$

$$\Gamma \geq g^*h^*(A) \geq \frac{1}{m} g^*h^*t^*(A_Z)$$

$$\Gamma \geq \frac{1}{m} \psi^*(A_Z)$$

ie $\Gamma = D + \frac{1}{m} \psi^*(A_Z)$

$$\kappa\left(\underbrace{K_Y + D + \frac{1}{m}\psi^*(A_Z)}\right) \geq \dim Z.$$

We can bound $\kappa(K_Y+\Gamma)$.

For $s \in \underbrace{H^0(\underline{m}(\underline{K_Y}+\Gamma))}$,

$$s \sim m\left(g^*(K_X+\Delta) - \underbrace{\varepsilon E' + E}\right)$$

$$g_* s \sim m\left(\underbrace{g_* g^*}(K_X+\Delta) + 0\right)$$

$$m(K_X+\Delta).$$

$$g_* s \in \underbrace{H^0(\underline{m}(K_X+\Delta))}$$

So $\underbrace{\kappa(K_X+\Delta)}_{0 \geq} \geq \underbrace{\kappa(K_Y+\Gamma)}_{\geq \dim Z}$.

$\Rightarrow \dim Z = 0$.

$\Rightarrow Z$ is a point.

Prop)

Let $f: X \to B$ be proper and B a smooth curve. s.t.

general fibre of f is rationally connected.

$\Rightarrow f$ has a section.

$\left\{ f: X \to S \text{ with r.c. fiber then we can lift rat. curves from } S \text{ to } X. \text{ (the lift is not a fiber).} \right\}$

Prop Maximal rationally connected fibration.

For X a variety. $\exists$

$\varphi : X \dashrightarrow Z$. characterized by

1) The general fiber is rationally connected.

2) For a very general point $z \in Z$. any rat. curve in X which intersects the fiber is contained in the fiber.

Lemma 2

Let F a normal variety.

1) F is rationally connected

iff $\forall\, t : F \dashrightarrow Z$ dominant non-constant non constant.

Z is uniruled.

2) F is rationally chain connected modulo V.

iff: $\forall\; T : F \dashrightarrow Z$ dominant rational

Either Z is uniruled or Z is dominated by V.

Proof

1) $\Rightarrow$ Z will also be rationally connected. Hence it is uniruled.

2) $\Rightarrow$ If V does not dominate

For a general point $z \in Z$.

we take preimage and it is not in V.. $\Rightarrow$ rational chain modulo V passes through it.

$\Rightarrow$ it is contained in a rat. curve.

1) $\Leftarrow$)

$id : F \to F$ we get that F is uniruled.

We take F' a smooth model of F. $t : F' \dashrightarrow Z$.

$t : F' \longrightarrow Z$ we can assume that t is a morphism.

So, we can lift $\mathbb{P}^1$'s.

If Z were uniruled, we would

have a rational curve through a general point.

We would lift that to a $\mathbb{P}^1$ in F' not in the fiber, but intersecting.

This contradicts MRC.

$\Rightarrow$ Z cannot be uniruled.

$\Rightarrow$ the map is constant.

$\Rightarrow$ F' is a fiber.

$\Rightarrow$ F' is r.c.

F is r.c.

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2) $\Leftarrow$ $t : F' \to Z$.

and we get that Z cannot be uniruled.

$\Rightarrow$ V dominates Z.

Then $x, y \in F$.

$\exists$ $v_1, v_2 \in V$ s.t.

$$\underbrace{t(v_1) = t(x)}_{\substack{\text{connected}\\ \text{by a } \mathbb{P}^1}}, \quad \underbrace{t(v_2) = t(y)}_{\substack{\text{connected}\\ \text{by a } \mathbb{P}^1}}$$

so, we get a r. chain modulo V through x and y.

$\Rightarrow F$ is r.c.c modulo V.

$\boxed{\text{Corollary}}$

Let (X, Δ) be a log pair, and $h : X \to F$.
Suppose that every rational map $t : F \dashrightarrow Z$.
either

$\boxed{1}$ The non-klt locus of

$(K_X+\Delta)$ dominates

or

2) $K_X+\Delta$ has Kodaira dim ≥ 0 on general fiber $X \dashrightarrow Z$

3) $\kappa(K_X+\Delta) \leq 0$

4) $\exists A$ ample s.t. $\pi^*A \leq \Delta$.

Then F is r.c.c modulo the image (R) of the non-klt locus of $(K_X+\Delta)$.

Proof: We need to check $\forall t: F \dashrightarrow Z$, either

R dominates Z or Z is uniruled.

We have that for $t: F \dashrightarrow Z$.

First case: R dominates Z.

2nd Case: R does not dominate Z. + 2) 3) 4).

These are the conditions for our first Lemma

$\Rightarrow$ Z is uniruled. ✓

The Lemma for r.c.c applies and we are done.

$\square$

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